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Determination of relative strain ellipsoids from sectional measurements of stretching lineation

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Abstract

A novel strain inversion method is developed in this communication to determine relative strain ellipsoid (namely, principal directions and relative magnitudes) from four or more independent measurements of stretching lineation or the longest elliptical axes on planar surfaces. A linear, five-variable equation is obtained to describe such kind of measurements. Equations for these measurements are solved under some auxiliary constraints for the relative strain ellipsoid. This is similar in formulation to stress inversion. We believe the method will provide for structural geologists a simple, useful and more applicable tool for estimating strain in deformed rock where no passive strain marker other than stretching lineation is commonly observed at outcrop.
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Keywords: Determination of finite strain; Relative strain ellipsoid; Sectional measurements; Stretching lineation

1. Introduction

For structural geologists, the determination of three-dimensional finite strain in rock is crucial in reconstructing the deformation history. Numerous graphic (e.g., Ramsay, 1967; Ramsay and Huber, 1983) and numerical (e.g., Shimamoto and Ikeda, 1976; Oertel, 1978; Milton, 1980; Gendzwill and Stauffer, 1981; Owens, 1984; Shao and Wang, 1984; Wheeler, 1989; De Paor, 1990; Robin, 2002; Shan et al., 2007) methods have been proposed for restoring the strain ellipsoid from strain ellipses measured on planar surfaces. In developing an inversion method recently for the same purpose, Shan et al. (2007) obtained a linear equation in six variables for each individual sectional measurement of stretching lineation or the longest elliptical axis, and highlighted an unresolved issue concerning the feasibility of using these equations to determine the strain ellipsoid. This issue is resolved in this short communication. As is proved below, this kind of strain measurement is sufficient for determining the relative strain ellipsoid (that is to say, principal directions and relative magnitude differences), but insufficient to determine absolute principal magnitudes.

A very similar idea can be derived from Tocher (1964) and others, who determined stereographically the optic axes of a crystal from a minimum of four independent extinction measurements, and was later applied to fabric analysis (Lisle, 1976). In essence, there is no difference in formulating sectional measurements of optic indicatrix, fabric ellipsoid and strain ellipsoid. However, a major contribution of this work is to render this concept in mathematics that will permit easy and fast use of a computer to make the determination.

Terms and their symbols used in this paper are listed in Table 1.

2. Presentation of the problem

Normally, a strain ellipsoid is considered in the Cartesian system (Fig. 1) as a quadric surface centered at the origin,

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Table 1
A list of symbols and their definitions

Symbols	Definitions	Comments
x, y, and z	Coordinates of a point on the ellipsoid in the real state.	Eq. (1).
X, Y, and Z	Initial coordinate system.	
x', y', and z'	Coordinates of a point on the ellipsoid in the rotated (reference) state.	Eqs. (3) and (4).
X', Y', and Z'	Rotated coordinate system.	
b_{ij}	Elements of a shape matrix.	i, j = 1,2,3; Eqs. (1), (4)–(6) and (8)–(11).
b_1, b_2, and b_3	Eigenvalues of a shape matrix.	$b_1 \geq b_2 \geq b_3 > 0$; Eq. (2).
ϵ_1, ϵ_2, and ϵ_3	Magnitudes of the principal axes of an ellipsoid.	$\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 > 0$; Eq. (2).
α and β	Dip direction (azimuth) and dip angle of a certain measured planar surface.	Eq. (3).
θ	Pitch of the long axis of a strain ellipse on the planar surface.	Eq. (3).
T	Inverse rotation matrix	Eq. (3).
t_{ij}	Elements of inverse rotation matrix.	i, j = 1,2,3; Eqs. (3)–(8) and (11).
k	Elliptical parameter	Eq. (4).

described by the following equation (e.g., Owens, 1984; Robin, 2002; Shan et al., 2007):

$$[x \quad y \quad z]\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \tag{1}$$

where x, y and z are the coordinates of a point on the ellipsoid in terms of the reference axes X, Y and Z, and b_{ij} ($i, j = 1, 2, 3$) is the element of the shape matrix (Shimamoto and Ikeda, 1976). The shape matrix is symmetrical, so $b_{ij} = b_{ji}$ ($i, j = 1, 2, 3$). The principal axes of the ellipsoid have the same directions as the eigenvectors of the shape matrix, but different dimensions from its eigenvalues,

$$\epsilon_1 = \frac{1}{\sqrt{b_3}}, \quad \epsilon_2 = \frac{1}{\sqrt{b_2}}, \quad \epsilon_3 = \frac{1}{\sqrt{b_1}} \tag{2}$$

where b_1, b_2, and b_3 are the corresponding eigenvalues of the shape matrix ($b_1 \geq b_2 \geq b_3 > 0$); ϵ_1, ϵ_2, and ϵ_3 are the corresponding principal radii of the ellipsoid ($\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 > 0$).

Consider measurement of the direction of the longest elliptical axis, or stretching lineation, on a planar surface. We measure on the surface the dip direction (α) and dip angle (β) of the surface, and the pitch (θ) of the long axis of the sectional ellipse of the 3-D ellipsoid. The pitch is defined as the intersection angle between the long axis of the ellipse and the westward strike of the plane after the plane has been rotated around a vertical axis to dip toward the northward X-axis (Fig. 1).

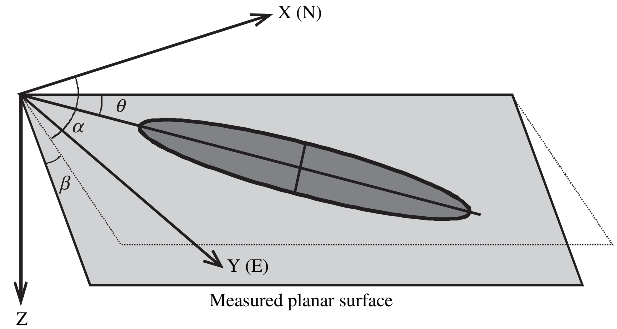

Fig. 1. Elements of strain measurement made on the planar surface in the Cartesian coordinate system (Shan et al., 2007). The X-axis is directed toward the north, the Y-axis toward the east, and the Z-axis upward. The blank rectangle, marked by three dashed lines and one thick line, represents a part of the horizontal (X–Y) plane. The gray rectangle marked by thick lines represents the plane where the stretching lineation is measured. See the text and Table 1 for symbol definitions.

For the given measurement, we will have a simple expression of the strain ellipse on the plane by manipulating a series of rotations to transform the strain measurement plane into a horizontal one where the long axis of the strain ellipse is aligned with the X-axis. This manipulation can be implemented by rotating around the Z-axis with an angle of $-\alpha$, around the Y-axis with an angle of $-\beta$, and finally around the Z-axis with an angle of $\theta - 90°$. Let T stand for the inverse manipulation of these rotations that defines the relationship between the old and the new coordinate systems:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{3}$$

$$\begin{aligned} T &= \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \\ &\quad \times \begin{bmatrix} \cos(90° - \theta) & \sin(90° - \theta) & 0 \\ -\sin(90° - \theta) & \cos(90° - \theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Application of the above transformation to Eq. (1) leads to the strain ellipsoid in its local X'–Y'–Z' coordinate system. Because the strain ellipse of interest lies in the X'–Y' plane after rotation, we have $Z' = 0$, thus giving the following expression for the rotated ellipse:

$$[x' \quad y']\begin{bmatrix} \begin{array}{l} t_{11}t_{11}b_{11} + 2t_{11}t_{21}b_{12} + 2t_{11}t_{31}b_{13} \\ +t_{21}t_{21}b_{22} + 2t_{21}t_{31}b_{23} + t_{31}t_{31}b_{33} \end{array} & \begin{array}{l} t_{11}t_{12}b_{11} + (t_{11}t_{22} + t_{21}t_{12})b_{12} \\ +(t_{11}t_{32} + t_{31}t_{12})b_{13} + t_{21}t_{22}b_{22} \\ +(t_{21}t_{32} + t_{31}t_{22})b_{23} + t_{31}t_{32}b_{33} \end{array} \\ (\text{symmetrical}) & \begin{array}{l} t_{12}t_{12}b_{11} + 2t_{12}t_{22}b_{12} + 2t_{12}t_{32}b_{13} \\ +t_{22}t_{22}b_{22} + 2t_{22}t_{32}b_{23} + t_{32}t_{32}b_{33} \end{array} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = k \tag{4}$$

where k is the elliptical parameter, depending upon each individual sectional measurement.

The local coordinate system is specified as having the section ellipse symmetrical to its axes, so the matrix in Eq. (4) must have zero for its off-diagonal elements. This is mandatory because, in the way described above, the rotated strain ellipse on the section has the largest and the smallest strain axis toward the X'- and the Y'-axis, respectively. Therefore, we have for this measurement the following linear equation.

$$t_{11}t_{12}b_{11}+(t_{11}t_{22}+t_{21}t_{12})b_{12}+(t_{11}t_{32}+t_{31}t_{12})b_{13}+t_{21}t_{22}b_{22} \\ +(t_{21}t_{32}+t_{31}t_{22})b_{23}+t_{31}t_{32}b_{33}=0 \tag{5}$$

It seems possible, according to Eq. (5), that we can determine the relative strain ellipsoid, that is to say the directions and the relative magnitudes of the strain principal axes, using only sectional measurements of this type. At least five such independent measurements would be needed, under some auxiliary constraint, such as the unit length of the unknown vector $[b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33}]$. Unfortunately, there are an infinite number of solutions of the unknown vector, no matter how many measurements are taken into consideration (Shan et al., 2007). This under-determinacy is attributed to intrinsic inadequacy in the determination of this type of strain measurement. Shan et al. (2007) noticed this problem, but failed to account for it in a strict sense.

3. Problem resolved

Eq. (5) can be rewritten in matrix form as follows:

$$\begin{bmatrix} t_{11} & t_{21} & t_{31} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} t_{12} \\ t_{22} \\ t_{32} \end{bmatrix} = 0 \tag{6}$$

By its definition, and as in Eq. (3), T is an orthogonal matrix. Accordingly, it consists of orthogonal columns. Take the first two columns, $[t_{11}, t_{21}, t_{31}]$ and $[t_{12}, t_{22}, t_{32}]$, for example. From them, we have

$$t_{11}t_{12}+t_{21}t_{22}+t_{31}t_{32}=0 \tag{7}$$

So, if the diagonal elements b_{11}, b_{22}, and b_{33} of the shape matrix have the same value, regardless of their magnitudes,

$$t_{11}t_{12}b_{11}+t_{21}t_{22}b_{22}+t_{31}t_{32}b_{33}=0 \tag{8}$$

These diagonal elements are therefore linearly dependent. That is why all the matrix elements cannot be determined simply from strain measurements of this kind. This is shown by solving for $t_{31}t_{32}$ in Eq. (7) and substituting into Eq. (5) to give the following equation:

$$t_{11}t_{12}(b_{11}-b_{33})+(t_{11}t_{22}+t_{21}t_{12})b_{12}+(t_{11}t_{32}+t_{31}t_{12})b_{13} \\ +t_{21}t_{22}(b_{22}-b_{33})+(t_{21}t_{32}+t_{31}t_{22})b_{23}=0 \tag{9}$$

As is apparent in the above equation, the three diagonal elements b_{11}, b_{22}, and b_{33} cannot simultaneously be determined, even with rescaling to unit length of the unknown vector $[b_{11}-b_{33}, b_{12}, b_{13}, b_{22}-b_{33}, b_{23}]$. The under-determinacy of the three diagonal elements lies in that only the differences $(b_{11}-b_{33})$ and $(b_{22}-b_{33})$ can be known in this way. To determine b_{11}, b_{22}, and b_{33} requires other kinds of strain measurements with more information, such as strain axial ratios.

Sensibly and justifiably, we may use a convention for specifying the diagonal elements by introducing the following auxiliary constraint:

$$b_{11}+b_{22}+b_{33}=0 \tag{10}$$

This is equivalent to determining the difference (hyperboloid) between the strain ellipsoid and a reference sphere of the equal volume.

Solving for b_{33} in Eq. (10) and substituting into Eq. (5) leads to the following equation:

$$(t_{11}t_{12}-t_{31}t_{32})b_{11}+(t_{11}t_{22}+t_{21}t_{12})b_{12}+(t_{11}t_{32}+t_{31}t_{12})b_{13} \\ +(t_{21}t_{22}-t_{31}t_{32})b_{22}+(t_{21}t_{32}+t_{31}t_{22})b_{23}=0 \tag{11}$$

So far, Eq. (11), as well as the constraint of unit length of the unknown vector $[b_{11}, b_{12}, b_{13}, b_{22}, b_{23}]$, constitutes the basis of our strain inversion method that determines the relative strain ellipsoid from measurements of the longest elliptical axes on the planar surfaces. A number of four or more independent measurements of this kind are needed for the determination. This is theoretically similar to inversion of stress from measured fault/slip data (Fry, 1999; Shan et al., 2003).

4. Procedure

In the previous section, we developed a new strain inversion method for determining the strain principal directions from sectional measurement of the longest elliptical axes or stretching lineation. The procedure to realize it is summarized as follows:

(1) For each strain measurement, calculate the rotation matrix T, according to Eq. (3),
(2) Calculate datum vector $[t_{11}t_{12}-t_{31}t_{32}, t_{11}t_{22}+t_{21}t_{12}, t_{11}t_{32}+t_{31}t_{21}, t_{21}t_{22}-t_{31}t_{32}, t_{21}t_{32}+t_{31}t_{32}]$ from the matrix T, according to Eq. (11),
(3) Rescale the datum vector to unit magnitude, for equal weight in the calculation to follow,
(4) Solve for the best solution of unknown vector $[b_{11}, b_{12}, b_{13}, b_{22}, b_{23}]$ through applying the moment method (Shan et al., 2003) to all unitized datum vectors,
(5) Calculate b_{33} from the solution, according to Eq. (10),
(6) Restore the shape matrix, according to Eq. (1), and
(7) Calculate the principal directions and relative magnitudes through applying the Jacobian method to the restored shape matrix.

For a set of such kind of artificial data having no measurement errors (Shan et al., 2007), estimated strain principal directions by using the proposed inversion method are similar to prescribed principal directions.

Table 2
Fabric data (Lisle, 1976) and results from using the progressive elimination method and the proposed method, respectively

Exposure planes (°)		Fabric traces (°)		Fabric ellipsoids (°)				
Dip direction	Dip angle	Bearing	Plunge	$\epsilon_1-\epsilon_2$ planes		Lineations		V
				Bearing	Plunge	Bearing	Plunge	
130	90	223	40	Lisle's (1975) result				
35	90	305	47	263	62	178	11	50.5
90	90	180	9	Our result				
181	90	271	64	265.0	63.6	179.4	8.7	41.4

Fabric traces are herein equivalent to stretching lineations observed on the exposure planes. V is the acute angle between two poles to planes showing circular sections in fabric ellipsoid. See the text and Lisle's (1976) paper for more explanation.

5. Application

The third example of Lisle (1976; see his Table 1) was used to demonstrate the feasibility of the method proposed above. It consists of four measurements that were made on a series of cozonal faces cut on a real hand specimen (Table 2). Lisle (1976) used the progressive elimination method (Tocher, 1964) in crystallography to determine the fabric in the specimens from these measurements (Table 2). Application of the proposed method to the example gives rise to the result listed in Table 2. It is very similar in the orientations of $\epsilon_1-\epsilon_2$ plane and of the lineation to the result of Lisle (1976). Only for the angle V, it turns out a great deal smaller, but still close to 43.5°, the lower limit by Lisle (1975).

Meanwhile, the eigenvalues obtained at step 4 are 5.63E−08, 2.29E−01, 3.80E−01, 8.43E−01, and 2.55, respectively. Compared with other eigenvalues, the least eigenvalue is extremely small, indicating that the solution of the unknown vector is well determined.

6. Conclusions and remarks

Sectional measurements of the longest elliptical axes or stretching lineation cannot directly be used as Eq. (5) to determine the strain principal ellipsoid, by virtue of their intrinsic inadequacy in the determination (Shan et al., 2007). This inadequacy lies in that the diagonal elements of the shape matrix, b_{11}, b_{22}, and b_{33}, are linearly dependent. To compensate for this requires the addition of an auxiliary constraint, as given in Eq. (9). Under this constraint, the shape matrix may be solved for using the moment method, just as in the inversion of stress from measured fault/slip data (Shan et al., 2003). The strain principal directions and relative magnitude differences, or the relative strain ellipsoid, are readily obtained from the calculated shape matrix. But the absolute principal magnitudes and their ratios remain unresolved, the determination of which needs other kinds of sectional measurements with more strain information (Shan et al., 2007).

Theoretically speaking, the proposed strain inversion method is very similar to the stress inversion method. Even without the record of axial ratios or the lengths of the elliptical axes, stretching lineation can be used to infer the relative strain ellipsoid. A minimal number of four such independent measurements is needed for this purpose. We believe it will provide for structural geologists a simple, useful and more applicable tool of estimating strain in deformed rock or rock fabrics where only stretching lineation is observed at outcrop – a common case that good passive strain markers are absent.

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